PROOFS OF SOME THEOREMS OF ELEMENTARY ALGEBRA BY THE METHOD OF MATHEMATICAL INDUCTION

Mirzakarimova Nigorakhon Mirzakhakimovna
Senior Lecturer, Fergana State University

Kholmatov Akhrorjon Khakimovich
Teacher, Academic Lyceum No. 2 under
Fergana State Technical University

Karimova Laylokhon, daughter of Dilshodbek
A student at Fergana State University

Theorem 1. The square of a polynomial with n terms is equal to the sum of the squares of all its terms plus twice the sum of all possible pairwise products of its terms.

$(a_1+a_2+\ldots+a_n)^2=a_1^2+a_2^2+\ldots+a_n^2+2(a_1a_2+a_1a_3+\ldots+a_{n-1}a)$ (1)

1. Formula (1) can be verified directly for $n=2$

$(a_1+a_2)^2=a_1^2+a_2^2+2a_1a_2$

2. Assume that formula (1) holds for $n=k-1$, that is,

$(a_1+a_2+\ldots+a_{n-1})^2=a_1^2+a_2^2+\ldots+a_{n-1}^2+2S$

where the sum is taken over all possible distinct pairwise products formed from $S-a_1,a_2,\ldots,a_{n-1}$

We must prove that the formula also holds for $n=k+1$,namely,

$(a_1+a_2+\ldots+a_{n-1}+a_k)^2=a_1^2+a_2^2+\ldots+a_{n-1}^2+a_n^2+2S$

where the sum includes all possible pairwise products formed from $S-a_1,a_2,\ldots,a_{n-1}$

$$S_1 = S + (a_1 + a_2 + \dots + a_{n-1})a_n$$

Indeed, expanding the expression confirms the statement.

$$(a_1 + a_2 + \dots + a_{n-1} + a_n)^2 = (a_1 + \dots + a_{n-1})^2 +$$

$$+2(a_1 + \dots + a_{n-1})a_n + a_n^2 = a_1^2 + a_2^2 + \dots + a_{n-1}^2 + 2S +$$

$$+2(a_1 + \dots + a_{n-1})a_n + a_n^2 = a_1^2 + a_2^2 + \dots + a_n^2 + 2S_1$$

Theorem 2. The n-th term of an arithmetic progression is given by the formula

$$a_n = a_1 + d(n-1) \qquad (1)$$

where a_1 – is the first term of the progression and d – is the common difference.

1. For $n=1$ formula (1) is correct.
2. Assume formula (1) holds for $n=k$. Then, $a_k = a_1 + d(k-1)$

Hence,

$$a_{k+1} = a_k + d = a_1 + d(k-1) + d = a_1 + d_k$$

Thus, formula (1) also holds for $n = k+1$.

Theorem 3. The n –th term of a geometric progression is calculated by the formula

$$a_n = a_1 q^{n-1}$$

where a_1 is the first term and q – is the common ratio.

1. The formula is true for $n=1$.
2. Assume it holds for $n=k$, that is,

$$a_n = a_1 q^{k-1}$$

Then, $a_{k+1} = a_k q = a_1 q^k$

Thus, the formula holds for $n = k+1$.

Theorem 4. The number of permutations of telements is determined by the formula

$P_t = t!$

1. First, note that for $t = 1$, the formula holds.
2. Assume it holds for $P_k = k!$.
3. We must prove it for $P_{k+1} = (k+1)!$.

Given $k+1$ elements, take the first k elements and construct all possible permutations of them. By assumption, the number of such permutations is k!.

Now, in each permutation, insert the element a_{k+1} into all possible positions: before the first element, before the second element, ..., before the k-th element, and finally at the end.

In this way, from each permutation of k elements we obtain $k+1$ permutations of $k+1$ elements.

Altogether, we obtain $k! \cdot (k+1) = (k+1)!$ permutations of $k+1$ elements.

We must verify the following:

1. Are there two identical permutations among these $(k+1)!$?
2. Have we obtained all permutations of the $k+1$ elements?

Suppose that among the constructed permutations $(k+1)!$there are two identical ones, denoted P_1 and P_2.

Assume that in permutation P_1, the element a_{k+1} occupies the s-th position from the left. Then in P_2, the element a_{k+1} must also occupy the s-th position.

Removing the element a_{k+1} from both P_1 and P_2, we obtain two identical permutations of the first k elements.

Removing the element $k+1$ from both P_1 and P_2 yields two identical permutations of the first k elements.Therefore, in order to obtain P_1 and P_2, the element $k+1$

must have been inserted twice into the same position of the same permutation of the first k elements.This contradicts the construction procedure for the permutations.Suppose, on the contrary, that there exists a permutation P of the $k+1$ elements that was not constructed.Assume that in P, the element a_{k+1} is in the s-th position from the left.Removing a_{k+1} from P, we obtain a permutation of the first k elements.Hence, to obtain $\bar{P}$, it suffices to take the corresponding permutation of the first k elements and insert the element a_{k+1} in the s-th position from the left.Such a permutation could not have been omitted, because all permutations of the first k elements had already been constructed.It was also impossible not to insert the element a_{k+1} into the specified position, since we had placed it in every possible position: first, second, ..., and $(k+1)$-th from the left. Therefore, all the permutations we constructed are distinct and consist of $k+1$ elements.From the above arguments, it follows that $P_{k+1} = (k+1)!$

Theorem 5. The number of arrangements (variations) of m elements taken n at a time is

$$A_m^n = m \cdot (m-1) \cdot \ldots \cdot (m-n+1) \qquad (1)$$

1. First of all, we would like to emphasize that $A_m = m$.
2. Letusassumethat $A_m^k = m \cdot (m-1) \cdot \ldots \cdot (m-k+1)$,

$$A_m^k = m \cdot (m-1) \cdot \ldots \cdot (m-k)$$

To generate all arrangements of m elements taken $k+1$ at a time, it is sufficient to take all arrangements of m elements taken k at a time and append to each of them one element from the remaining $m-k$ elements. It is easy to see that all arrangements of m elements taken $k+1$ at a time constructed in this way are

distinct from each other, and moreover, any arrangement of m elements taken $k+1$ at a time will appear among those generated.

Therefore, this conclusion follows $A_m^{k+1} = A_m^k(m-k) = m(m-1)\cdot .....\cdot (m-k)$

Theorem 6. The number of combinations of m elements taken n at a time is

$$C_m^n = \frac{m(m-1)\cdot .....\cdot (m-n+1)}{1\cdot 2\cdot ...\cdot n} \qquad (1)$$

1. Assume the formula holds for $C'_m = m$.
2. Letusassumethat $C_m^k = \dfrac{m(m-1)\cdot ...\cdot (m-n+1)}{1\cdot 2\cdot ...\cdot k}$
3. We willprovethat $C_m^{k+1} = \dfrac{m(m-1)\cdot ...\cdot (m-n+1)\cdot (m-k)}{1\cdot 2\cdot ...\cdot k\cdot (k+1)}$

To generate all combinations of m elements taken $k+1$ at a time, we first list all combinations of m elements taken k at a time and then, for each of these, sequentially add each of the remaining $m-k$ elements. Clearly, all combinations of m elements taken $k+1$ at a time are obtained in this way, but each of them appears exactly $k+1$ times. Indeed, a combination: $a_1, a_2, ..., a_k, a_{k+1}$

1. When element a_1 is added to the combination $a_1, a_2, ..., a_k, a_{k+1}$
2. When element a_2 is added to the combination, and so on $a_1, a_2, ..., a_k, a_{k+1}$
3. Finally, when element a_{k+1} is added to the combination, it is obtained $a_1, a_2, ...a_k$

Thus

$$C_m^{k+1} = \frac{C_m^k \cdot m(m-k)}{k+1} = \frac{m(m-1)\cdot ...\cdot (m-k)}{1\cdot 2\cdot ...\cdot k\cdot (k+1)}$$

Theorem 7 (Binomial Theorem). For any numbers a and b, and for any natural number n, the following formula holds:

$$(a+b)^n = a^n C_n^1 a^{n-1} b + \ldots + C_n^s q^{n-s} b^s + \ldots + C_n^{n-1} a b^{n-1} + b^n \qquad (1)$$

1. For $n=1$, the formula is clearly true.
2. Assume it holds for

$$(a+b)^k = a^k + C_k^1 a^{k+1} b + C_k^2 a^{k-2} b^2 + \ldots + b^k$$

$$n = k+1, \quad (a+b)^{k+1} = (a+b)^k (a+b) = (a^k + C_k^1 a^{k+1} b + \ldots + b^k)(a+b) =$$

$$= a^k + (1 + C_k^1) a^k b + (C_k^1 + C_k^2) \cdot a^{k-1} b^2 + \ldots + b^{k+1}$$

$$C_k^3 + C_l^{s+1} = C_{k+1}^{s+1}.$$

$$(a+b)^{k+1} = a^{k+1} + C_{k+1}^1 a^k b + C_{k+1}^2 a^k b + C_{k+1}^2 a^{k-1} b^2 + \ldots + C_{k+1}^{s+1} a^{k-s} b^{s+1} + \ldots + b^{k+1}$$

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